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# Stagnation flow with a temperature-dependent viscosity

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In this paper we derive solutions for two stagnation flows of an incompressible Newtonian fluid with infinite Prandtl number and exponentially temperature-dependent viscosity. The two stagnation flows are the impingement of a hot fluid against a cold wall and against a cold half-space of the same material. We find that the same solutions apply to both axisymmetric and two-dimensional flows. We apply these solutions to the thinning of the Earth's lithosphere by a mantle plume. The equilibrium lithospheric thickness and the rate of lithospheric thinning are obtained.

## 1. Introduction

Fluids with a temperature-dependent rheology are important in a number of applications. Examples include fabrication processes involving polymers and glasses. A temperature-dependent rheology is also appropriate for the solid-state creep of the Earth's mantle on geological timescales. In this paper we derive solutions for two stagnation flows with temperature-dependent viscosities. Our analysis is restricted to incompressible Newtonian fluids with infinite Prandtl number and exponentially temperature-dependent viscosity.

The first stagnation flow is the impingement of a hot fluid against a cold wall. We obtain the thickness of the thermal boundary layer and the heat flux from the fluid to the wall. Figure 1 (*a*) illustrates the first stagnation flow. The second stagnation flow is the impingement of a hot fluid against a half-space of the same material. The half-space is sufficiently cold that it behaves as a rigid body. The hot fluid continuously heats and entrains the cold half-space, and we solve for the rate of destruction of the cold half-space by the hot stagnation-point flow. Related problems are the impingement of a fluid phase upon a solid phase (Roberts 1958; Turcotte 1960) and laminar-flame propagation in premixed gases (e.g. Penner 1957). We first derive solutions for axisymmetric stagnation flows and then show that the same solutions apply to two-dimensional stagnation flows.

## 2. Cold-wall boundary condition

Landau & Lifshitz (1959) give the governing equations of an incompressible Newtonian fluid with infinite Prandtl number and temperature-dependent viscosity. Conservation of momentum, mass and energy for axisymmetric flows requires

$$0=\frac{-1}{\rho}\frac{\partial p}{\partial r}+\nu\left(\frac{\partial^2 v_r}{\partial r^2}+\frac{\partial^2 v_z}{\partial z^2}+\frac{1}{r}\frac{\partial v_r}{\partial r}-\frac{v_r}{r^2}\right)+\frac{d\nu}{dT}\left(2\frac{\partial v_r}{\partial r}\frac{\partial T}{\partial r}+\frac{\partial v_z}{\partial r}\frac{\partial T}{\partial z}+\frac{\partial v_r}{\partial z}\frac{\partial T}{\partial z}\right), \quad (1)$$

$$0=\frac{-1}{\rho}\frac{\partial p}{\partial z}+\nu\left(\frac{\partial^2 v_z}{\partial r^2}+\frac{\partial^2 v_z}{\partial z^2}+\frac{1}{r}\frac{\partial v_z}{\partial r}\right)+\frac{d\nu}{dT}\left(2\frac{\partial v_z}{\partial z}\frac{\partial T}{\partial z}+\frac{\partial v_r}{\partial z}\frac{\partial T}{\partial r}+\frac{\partial v_z}{\partial r}\frac{\partial T}{\partial r}\right), \quad (2)$$

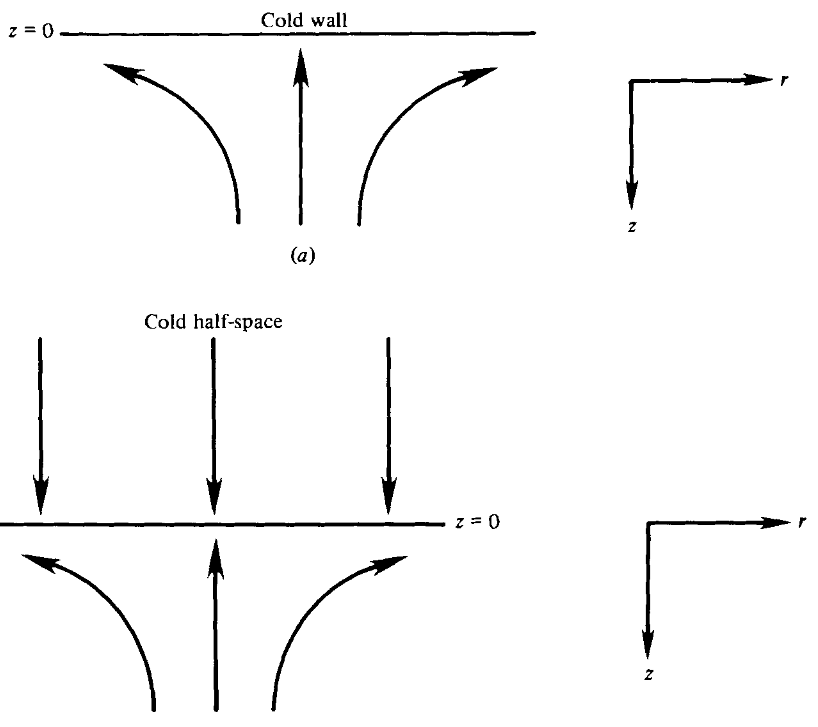


FIGURE 1. (a) Hot stagnation flow with a cold wall boundary condition. (b) Hot stagnation flow with a cold-half-space boundary condition. The fluid and the half-space are composed of the same material. The hot fluid continuously heats and entrains the cold half-space.

$$\frac{\partial v_r}{\partial r}+\frac{\partial v_z}{\partial z}+\frac{v_r}{r}=0, \tag{3}$$

$$\frac{\partial T}{\partial t}+v_r\frac{\partial T}{\partial r}+v_z\frac{\partial T}{\partial z}=\kappa\left(\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{\partial^2 T}{\partial z^2}\right), \tag{4}$$

where $p$ is pressure, $v_r$ is radial velocity, $v_z$ is vertical velocity, $T$ is temperature, $\rho$ is density, $\nu$ is kinematic viscosity and $\kappa$ is thermal diffusivity.

In the vicinity of a stagnation point it is appropriate to assume

$$v_z=-2f(z), \quad v_r=r\frac{df}{dz}, \quad T=T(z) \tag{5}$$

(Roberts 1958; Schlichting 1979). The velocity components given in (5) satisfy conservation of mass, and substitution of (5) into (1), (2) and (4) yields

$$0=\frac{-1}{\rho r}\frac{\partial p}{\partial r}+\frac{d}{dz}\left(\nu\frac{d^2 f}{dz^2}\right), \tag{6}$$

$$0=\frac{-1}{\rho}\frac{\partial p}{\partial z}-2\nu\frac{d^2 f}{dz^2}-4\frac{df}{dz}\frac{d\nu}{dT}\frac{dT}{dz}, \tag{7}$$

$$-2f(z)\frac{dT}{dz}=\kappa\frac{d^2 T}{dz^2}. \tag{8}$$

Equation (7) implies $\partial^2 p/\partial r\,\partial z = 0$. Hence, differentiating (6) with respect to $z$ we obtain

$$\frac{d^2}{dz^2}\left(\nu\frac{d^2 f}{dz^2}\right) = 0. \tag{9}$$

Note that adding a buoyancy term or a gravity term to (2) would leave equation (9) for $f(z)$ unchanged.

At the cold wall, we require no-slip boundary conditions. Far from the wall, the flow is isothermal. If the flow is isothermal, then (9) implies that $f(z)$ is a cubic polynomial. Since $f(0) = df(0)/dz = 0$, the leading term in the polynomial is the quadratic term. Hence the boundary conditions are

$$f = \frac{df}{dz} = 0, \quad T = T_0 \quad (z = 0), \tag{10a}$$

$$\frac{d^2 f}{dz^2} = a, \qquad T = T_\infty \quad (z = \infty). \tag{10b}$$

Utilizing (10), we obtain an inner solution that can be matched to a variety of outer solutions. Examples of outer solutions include a Stokes flow over a sphere and the impingement of a thermal plume upon a flat plate. We will consider the latter problem in some detail in §4 in terms of the impingement of a mantle plume on the Earth's lithosphere.

Now we specialize to an exponentially temperature-dependent viscosity

$$\nu = \nu_0 \exp(-T/T_a) \tag{11}$$

and define dimensionless variables and parameters

$$\phi = (\kappa^2 a)^{-\frac{1}{3}} f, \quad \eta = \left(\frac{a}{\kappa}\right)^{\frac{1}{3}} z, \quad \theta = \frac{T - T_0}{T_\infty - T_0}, \quad b = \frac{T_\infty - T_0}{T_a}. \tag{12}$$

Inserting (11) and (12) into (8)–(10) yields

$$\frac{d^2\theta}{d\eta^2} + 2\phi\frac{d\theta}{d\eta} = 0, \tag{13}$$

$$\frac{d^2}{d\eta^2}\left[\exp(-b\theta)\frac{d^2\phi}{d\eta^2}\right] = 0, \tag{14}$$

with boundary conditions

$$\phi = \frac{d\phi}{d\eta} = \theta = 0 \quad (\eta = 0), \tag{15a}$$

$$\frac{d^2\phi}{d\eta^2} = \theta = 1 \quad (\eta = \infty). \tag{15b}$$

Note that, since (9) is a homogeneous equation, the pre-exponential factor in the viscosity law does not enter into the analysis. Integrating (14) together with boundary conditions (15), we obtain

$$\exp(-b\theta)\frac{d^2\phi}{d\eta^2} = \exp(-b). \tag{16}$$

Combining (13,), (15) and (16) gives

$$\frac{d^2\phi}{d\eta^2}\frac{d^4\phi}{d\eta^4} - \left(\frac{d^3\phi}{d\eta^3}\right)^2 + 2\phi\frac{d^2\phi}{d\eta^2}\frac{d^3\phi}{d\eta^3} = 0, \tag{17}$$

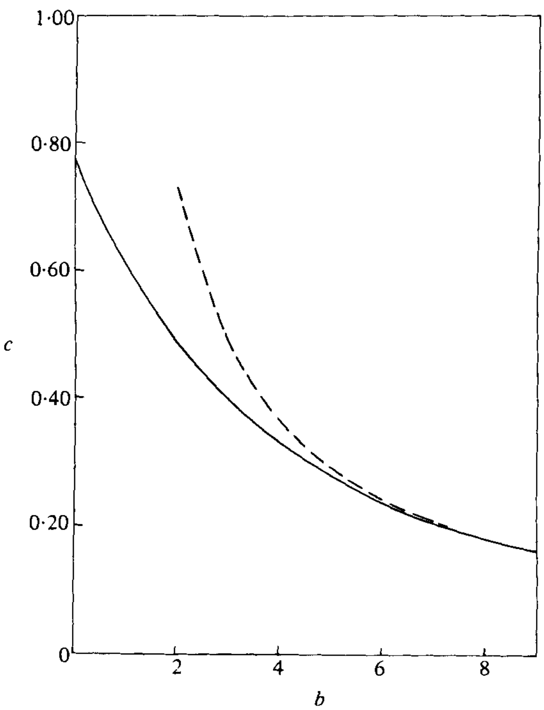


FIGURE 2. Dimensionless stagnation heat flux $c$ as a function of $b$ for the cold-wall boundary condition. The solid curve is calculated from (17). The dashed curve is $c = 1{\cdot}46/b$. For $b > 8$ the two curves are indistinguishable.

with boundary conditions

$$\phi = \frac{d\phi}{d\eta} = 0, \quad \frac{d^2\phi}{d\eta^2} = \exp(-b) \quad (\eta = 0), \tag{18a}$$

$$\frac{d^2\phi}{d\eta^2} = 1 \quad (\eta = \infty). \tag{18b}$$

Equation (17) is a fourth-order nonlinear ordinary differential equation. It can be solved analytically only in the case $b = 0$ (the constant-viscosity case), when we find

$$\phi = \tfrac{1}{2}\eta^2, \tag{19}$$

$$\theta = \frac{3^{\frac{2}{3}}}{\Gamma(\frac{1}{3})} \int_0^\eta \exp(-\tfrac{1}{3}\xi^3)\, d\xi, \tag{20}$$

where $\Gamma$ is the gamma function. We solve (17) and (18) numerically by guessing $d^3\phi/d\eta^3$ at $\eta = 0$ and integrating to find $d^2\phi/d\eta^2$ at infinity, i.e. at large $\eta$. We then iterate on $d^3\phi(0)/d\eta^3$ until we obtain $d^2\phi(\infty)/d\eta^2 = 1$. From (16) we obtain the dimensionless stagnation heat flux

$$c = \frac{d\theta(0)}{d\eta} = \frac{1}{b}\exp(b)\frac{d^3\phi(0)}{d\eta^3}.$$

Figure 2 gives $c$ as a function of $b$. For $b > 8$, $c = 1{\cdot}46/b$ is an excellent approximation. Figures 3 and 4 show $\phi(\eta)$ and $\theta(\eta)$ for various values of $b$. Figure 5 gives the thermal boundary-layer thickness as a function of $b$, where the thermal boundary layer is defined as that portion of the fluid for which $0 < \theta < 0{\cdot}9$.

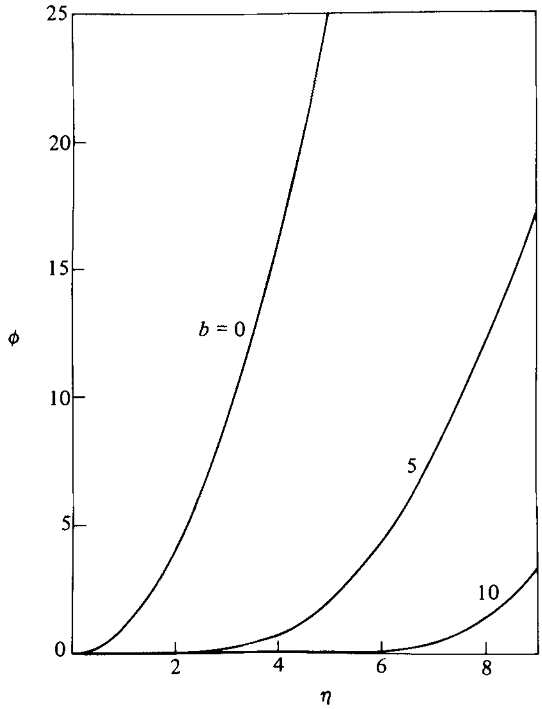


FIGURE 3. Dimensionless velocity $\phi$ as a function of dimensionless distance $\eta$ for the cold-wall boundary condition.

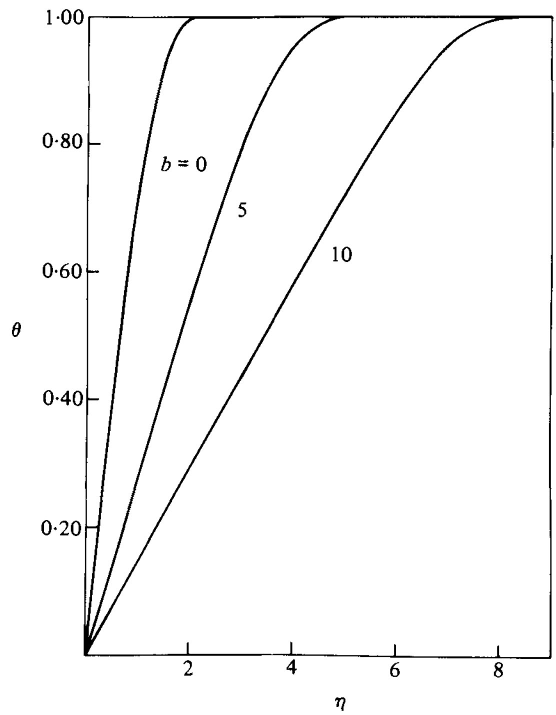


FIGURE 4. Dimensionless temperature $\theta$ as a function of dimensionless distance $\eta$ for the cold-wall boundary condition.

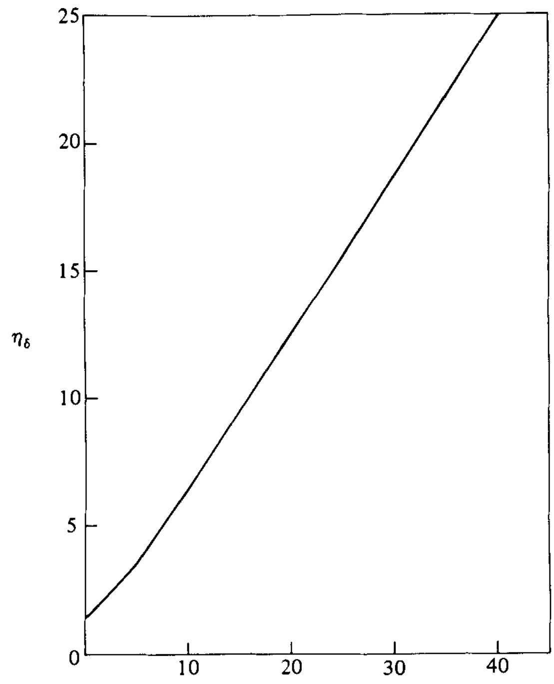


FIGURE 5. Dimensionless thermal boundary-layer thickness as a function of $b$ for the cold-wall boundary condition. The thermal boundary layer is that portion of the fluid for which $0 < \theta < 0{\cdot}9$.

If $b$ is sufficiently large, then $d^2\phi(0)/d\eta^2 = \exp(-b)$ will exceed a computer's underflow limit. We remedy this problem by expanding the solution of (13) and (16) in powers of $\epsilon = \exp(-b)$, and then use this expansion to obtain $\phi$, $d\phi/d\eta$, $d^2\phi/d\eta^2$ and $d^3\phi/d\eta^3$ at some $\eta$ where $d^2\phi/d\eta^2$ is sufficiently large that (17) can be solved on a computer. Let

$$\theta = c\eta + \epsilon\tilde{\theta}, \quad \phi = \epsilon\tilde{\phi}. \tag{21}$$

Inserting (21) into (13) and (16) and equating powers of $\epsilon$ gives

$$\exp(-bc\eta)\frac{d^2\tilde{\phi}}{d\eta^2} = 1, \tag{22}$$

$$2c\tilde{\phi} + \frac{d^2\tilde{\theta}}{d\eta^2} = 0. \tag{23}$$

Solving (22) and (23) together with the boundary conditions

$$\tilde{\theta}(0) = \frac{d\tilde{\theta}(0)}{d\eta} = \tilde{\phi}(0) = \frac{d\tilde{\phi}(0)}{d\eta} = 0,$$

we obtain

$$\phi = \exp(-b)\left[\frac{1}{b^2c^2}\exp(bc\eta) - \frac{1}{bc}\eta - \frac{1}{b^2c^2}\right],$$

$$\theta = c\eta + \exp(-b)\left[\frac{-2}{b^4c^3}\exp(bc\eta) + \frac{1}{3b}\eta^3 + \frac{1}{b^2c}\eta^2 + \frac{2}{b^3c^2}\eta + \frac{2}{b^4c^3}\right]. \tag{25}$$

The same solutions that apply to axisymmetric stagnation flows also apply to two-dimensional stagnation flows. In Cartesian coordinates the governing equations of two-dimensional incompressible Newtonian flow with infinite Prandtl number and temperature-dependent viscosity are (Landau & Lifschitz 1959)

$$0 = \frac{-1}{\rho}\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2 v_x}{\partial x^2}+\frac{\partial^2 v_x}{\partial z^2}\right)+\frac{d\nu}{dT}\left(2\frac{\partial v_x}{\partial x}\frac{\partial T}{\partial x}+\frac{\partial v_z}{\partial x}\frac{\partial T}{\partial z}+\frac{\partial v_x}{\partial z}\frac{\partial T}{\partial z}\right), \tag{26}$$

$$0 = \frac{-1}{\rho}\frac{\partial p}{\partial z}+\nu\left(\frac{\partial^2 v_z}{\partial x^2}+\frac{\partial^2 v_z}{\partial z^2}\right)+\frac{d\nu}{dT}\left(2\frac{\partial v_z}{\partial z}\frac{\partial T}{\partial z}+\frac{\partial v_x}{\partial z}\frac{\partial T}{\partial x}+\frac{\partial v_z}{\partial x}\frac{\partial T}{\partial x}\right), \tag{27}$$

$$\frac{\partial v_x}{\partial x}+\frac{\partial v_z}{\partial z} = 0, \tag{28}$$

$$\frac{\partial T}{\partial t}+v_x\frac{\partial T}{\partial x}+v_z\frac{\partial T}{\partial z} = \kappa\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial z^2}\right), \tag{29}$$

where $v_x$ is horizontal velocity. Letting

$$v_z = -2f(z), \quad v_x = 2x\frac{df}{dz}, \quad T = T(z), \tag{30}$$

we obtain

$$\frac{d^2}{dz^2}\left(\nu\frac{d^2 f}{dz^2}\right) = 0, \tag{31}$$

$$-2f(z)\frac{dT}{dz} = \kappa\frac{d^2T}{dz^2}, \tag{32}$$

which are identical to (9) and (8) respectively. We then follow the same procedure as before to obtain (17).

## 3. Cold half-space boundary condition

We next solve for the rate of destruction of a cold half-space of ambient temperature $\theta = 0$ by a hot stagnation flow of ambient temperature $\theta = 1$. The stagnation flow and the half-space are composed of the same material. The stagnation flow heats the half-space, thus incorporating the half-space into the stagnation flow. Consider an isotherm $\theta = \theta^*$ that is well within the cold half-space. We assume that for $\theta \leqslant \theta^*$ the half-space is sufficiently cold that it can be regarded as rigid. Of course, for any finite temperature the half-space will not actually be rigid. But in order to make the problem well-defined it is necesary to assume that the viscosity is infinite for $\theta$ less than some $\theta^*$. In the related problem of laminar-flame propagation it is assumed that the reaction rate is zero at a sufficiently low temperature (Penner 1957). This is a reasonable assumption as long as finite upstream regions are considered.

We fix the isotherm $\theta = \theta^*$ at $\eta = 0$ and solve for the velocity $\phi_0$ of the half-space at $\eta = 0$. For $\theta \leqslant \theta^*$ the half-space is sufficiently cold so that radial velocity vanishes for $\eta \leqslant 0$. By solving (13) with constant $\phi = \phi_0$ and boundary conditions $\theta(0) = \theta^*$ and $\theta(-\infty) = 0$, we can show that $\phi_0 = -c/2\theta^*$, where $c$ is the dimensionless stagnation heat flux. Hence, we find the rate of destruction $\phi_0$ of a half-space by solving (17) with boundary conditions

$$\left.\begin{aligned} \phi = \phi_0, \quad \frac{d\phi}{d\eta} = 0, \quad \frac{d^2\phi}{d\eta^2} = \exp[-b(1-\theta^*)], \\ \frac{d^3\phi}{d\eta^3} = -2\theta^*\phi_0 b\exp[-b(1-\theta^*)] \end{aligned}\right\} \quad (\eta = 0), \tag{33a}$$

$$\frac{d^2\phi}{d\eta^2} = 1 \quad (\eta = \infty). \tag{33b}$$

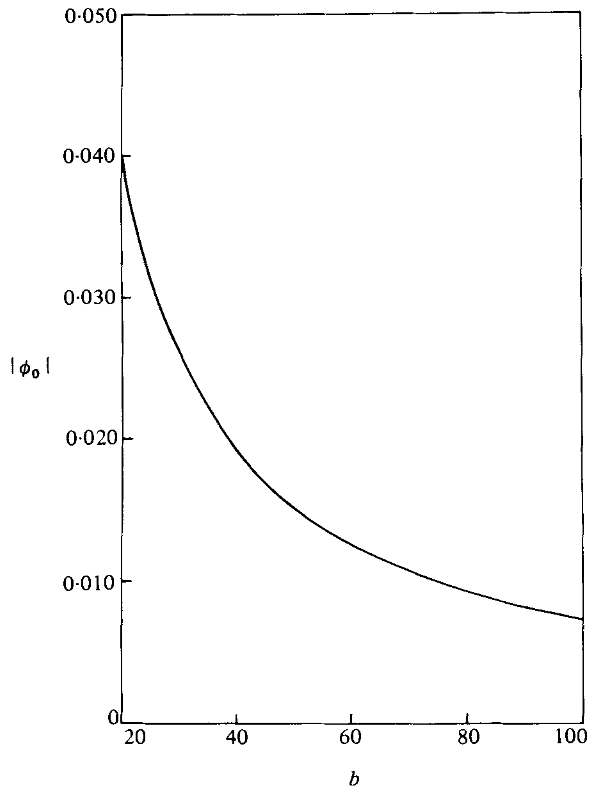


FIGURE 6. Dimensionless rate of destruction $|\phi_0|$ of a cold half-space by a hot stagnation flow as a function of $b$.

The boundary conditions on $d^2\phi/d\eta^2$ and $d^3\phi/d\eta^3$ at $\eta = 0$ follow from $\phi_0 = -c/2\theta^*$ and (13). We choose a $\theta^*$ that is deep in the rigid half-space, guess $\phi_0$, and integrate to obtain $d^2\phi/d\eta^2$ at infinity. We then iterate on $\phi_0$ until we obtain $d^2\phi/d\eta^2 = 1$. Then we take $\theta^*$ to be a fraction of its previous value and again solve for $\phi_0$. If $b > 20$ and $\theta^*$ is sufficiently small, then $\phi_0$ is unchanged. However, if $b < 20$, $\phi_0$ does change, whatever the value of $\theta^*$. That is, the viscosity contrast between the stagnation flow and the half-space is not sufficiently great so that the half-space can be regarded as rigid. Figure 6 shows $|\phi_0|$ (note that $\phi_0$ is negative) as a function of $b$ for $b > 20$. Figures 7 and 8 show $\phi(\eta)$ and $\theta(\eta)$ for $b = 20$.

## 4. Geophysical applications

We conclude with some applications of the two stagnation flows to convection in the Earth's mantle. Although this paper considers viscosity laws only of the form (11), the temperature dependence of the viscosity of the Earth's mantle is generally regarded as having the form

$$\nu \sim \exp(T^*/T) \tag{34}$$

(e.g. Oxburgh & Turcotte 1978). Kohlstedt, Goetze & Durham (1976) give $T^* = 6{\cdot}2928 \times 10^4$ K. However, geophysicists often use (11) as an approximation to (34) (McKenzie 1977; McKenzie & Weiss 1980; Brun & Cobbold 1980; Morris 1981).

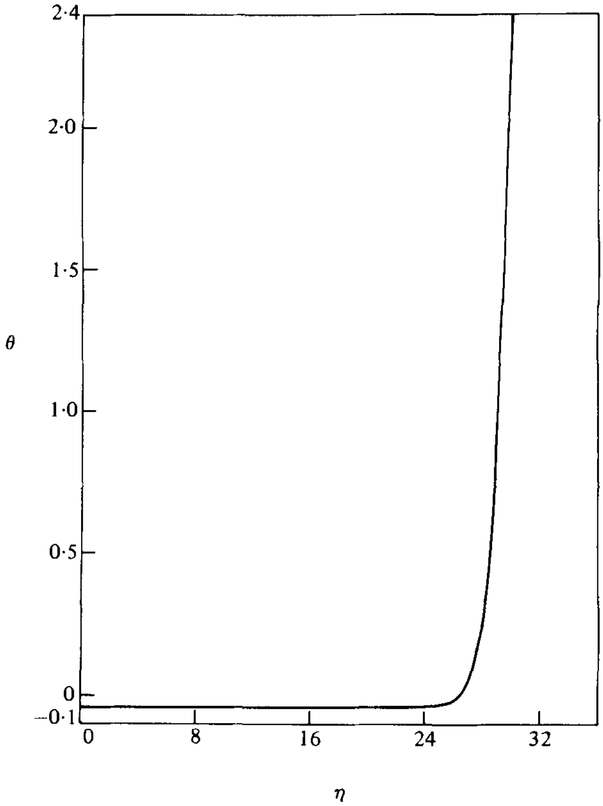


FIGURE 7. Dimensionless velocity $\phi$ as a function of dimensionless distance $\eta$ for the cold-half-space boundary condition. The origin $\eta = 0$ corresponds to the isotherm $\theta = 0{\cdot}1$. Note that $\phi_0$, the velocity of the half-space, is negative. $b = 20$.

The approximation was first used by Frank-Kamenetsky (1939) and is a good approximation as long as the temperature range is small and/or $T^*$ is large.

In order to utilize our analysis it is necessary to specify the stagnation-flow parameter $a$. No one has published a numerical simulation of convection using a viscosity law of the form (11). However, Parmentier, Turcotte & Torrance (1975) have done a numerical simulation of axisymmetric mantle convection using a viscosity law $\nu \sim \exp[50/(1+\frac{1}{2}\theta)]$. Since most of the flow takes place close to $\theta = 1$, we expand the exponent about $\theta = 1$ to obtain $\nu \sim \exp(-11{\cdot}1\theta)$. We disregard the pre-exponential factors since, as mentioned above, they do not enter the analysis. Parmentier *et al.* (1975) found a maximum surface heat flux of $3{\cdot}5 \times 10^{-6}$ cal cm$^{-2}$ s$^{-1}$ for a thermal diffusivity of $\kappa = 10^{-2}$ cm$^2$ s$^{-1}$, a thermal conductivity of $k = 9{\cdot}9 \times 10^{-3}$ cal K$^{-1}$ cm$^{-1}$ s$^{-1}$, $T_\infty = 2100$ K and $T_0 = 1400$ K. Using figure 2 and (12), we find $a = 5{\cdot}68 \times 10^{-19}$ cm$^{-1}$ s$^{-1}$.

A geophysical application of the first stagnation flow (cold-wall boundary condition) is the impingement of a cylindrical convective mantle plume against the Earth's surface (Morgan 1971). The Earth's surface is sufficiently cold that we can treat it as a rigid wall with respect to the Earth's mantle. We wish to determine the thickness of the thermal boundary layer, or lithosphere, above a mantle plume. Using figure 5, the above value of $a$, the above parameters except that now $T_0 = 273$ K, and (12), we find the thickness of the lithosphere above a mantle plume to be 41 km.

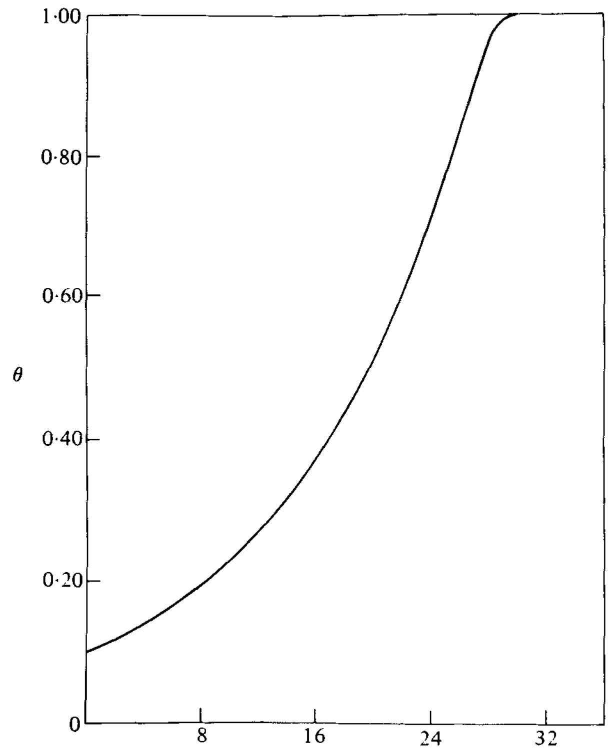


FIGURE 8. Dimensionless temperature $\theta$ as a function of dimensionless distance $\eta$ for the cold-half-space boundary condition. The origin $\eta = 0$ corresponds to the isotherm $\theta = 0{\cdot}1$. $b = 20$.

Comparisons of anomalous topography and gravity adjacent to Hawaii and other sites of intraplate volcanism give lithospheric thicknesses of about 50 km (Crough 1981), in good agreement with our theoretical calculation.

A geophysical application of the second stagnation flow (cold-half-space boundary condition) is the thinning of the lithosphere by a mantle plume, a process thought to be responsible for the formation of midplate swells (Crough 1978). Using figure 6, the above value of $a$, the above parameters except that now $T_0 = 700$ K, and (12), we find a thinning rate of 0·995 km Ma$^{-1}$. The thinning rate we find is an order of magnitude smaller than is required by observations (Detrick & Crough 1978). From (12) we see that we can increase the thinning rate by an order of magnitude only by increasing the value of $a$ by three orders of magnitude. But from (10) we see that increasing $a$ by three orders of magnitude also increases the ambient plume velocity by three orders of magnitude, and mantle plume velocities of kilometres per year are unreasonable. This raises doubts about the ability of mantle plumes to thin the lithosphere without the aid of partial melting, hydrofracturing, or other effects (Withjack 1979; Turcotte 1981).

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